



Light transport in homogeneous tissue with m-dependent anisotropic scattering II: Fourier transform solution and consistent relations

LIN WANG, MENG RONG, AND KAIYANG LI*

Wuhan University, School of Physics and Technology, Department of Electronic Science and Technology, No. 299, Bayi Road, Wuchang District, Wuhan City, Hubei Province 430072, China

*lky@whu.edu.cn

Abstract: This paper is the second of two focusing on the analytical solutions for light transport in infinite homogeneous tissue with an azimuth-dependent (m -dependent) anisotropic scattering kernel by two approaches, Case's singular eigenfuncions (CSEs) expansion and Fourier transform, and proving the consistence of the two solutions theoretically. In this paper, the analytical solution for the m -dependent truncated scattering kernel was derived via the Fourier transform and inversion, and expanded with the m -dependent generalized singular eigenfuncions (GSEs). Two kinds of GSEs that are defined by Ganapol in the case $m=0$ are extended to arbitrary azimuthal orders and proven to be consistent with CSEs both in expression forms and in intrinsic behaviors. By applying the Fourier transform inversion on the solution for the three-term recurrences, the Green's function of radiance distributions is obtained successfully, and it conforms perfectly to the CSEs solution in the limit, which has already been discussed in our first accompanying paper. Meanwhile, as a byproduct, a series of identities about the m -dependent Chandrasekhar orthogonal polynomials were presented and will be greatly helpful for further studies.

© 2018 Optical Society of America under the terms of the [OSA Open Access Publishing Agreement](#)

OCIS codes: (170.3660) Light propagation in tissues; (290.5825) Scattering theory; (170.7050) Turbid media.

References and links

1. A. D. Kim and J. B. Keller, "Light propagation in biological tissue," *J. Opt. Soc. Am. A* **20**(1), 92–98 (2003).
2. A. D. Kim, "Transport theory for light propagation in tissues," in *Biomedical Topical Meeting*, OSA Technical Digest (Optical Society of America, 2004), paper SD4.
3. L. V. Wang and H. I. Wu, *Biomedical Optics: Principles and Imaging*, (Wiley-Interscience, 2007).
4. A. J. Welch and M. J. C. V. Gemert, *Optical-Thermal Response of Laser-Irradiated Tissue*, (Springer Netherlands, 2011).
5. A. Liemert and A. Kienle, "Light transport in three-dimensional semi-infinite scattering media," *J. Opt. Soc. Am. A* **29**(7), 1475–1481 (2012).
6. V. V. Tuchin, "Light scattering study of tissues," *Phys-USP* **40**(5), 495–515 (1997).
7. M. Machida, "Singular eigenfunctions for the three-dimensional radiative transport equation," *J. Opt. Soc. Am. A* **31**(1), 67–74 (2014).
8. A. D. Klose and E. W. Larsen, "Light transport in biological tissue based on the simplified spherical harmonics equations," *J. Comput. Phys.* **220**(1), 441–470 (2006).
9. P. González-Rodríguez and A. D. Kim, "Light propagation in tissues with forward-peaked and large-angle scattering," *Appl. Opt.* **47**(14), 2599–2609 (2008).
10. K. M. Case and R. D. Hazeltine, "Fourier transform methods in linear transport theory," *J. Math. Phys.* **12**(9), 1970–1980 (1971).
11. K. Kobayashi, H. Kikuchi, and K. Tsutsuguchi, "Solution of multigroup transport equation in x-y-z geometry by the spherical harmonics method using finite fourier transformation," *J. Nucl. Sci. Technol.* **30**(1), 31–47 (1993).
12. B. D. Ganapol, "A consistent theory of neutral particle transport in an infinite medium," *Transp. Theory Stat. Phys.* **29**(1–2), 43–68 (2000).
13. B. D. Ganapol, "Fourier transform transport solutions in spherical geometry," *Transp. Theory Stat. Phys.* **32**(5–7), 587–605 (2003).
14. B. D. Ganapol, "Chandrasekhar polynomials and the solution to the transport equation in an infinite medium," *J. Comput. Theor. Trans.* **41** (1–7), 433–473 (2014).
15. B. D. Ganapol, "The Fourier transform solution for the Green's function of monoenergetic neutron transport theory," in *2014 Hawaii University International Conferences on Science, Technology, Engineering, Math & Education* (2014).

16. B. D. Ganapol, "The infinite medium Green's function of monoenergetic neutron transport theory via Fourier transform," *Nucl. Sci. Eng.* **180**(2), 224–246 (2015).
17. M. Machida, "The Green's function for the three-dimensional linear Boltzmann equation via Fourier transform," *J. Phys. A Math. Theor.* **49**(17), 175001 (2016).
18. İnönü, and Erdal, "Orthogonality of a set of polynomials encountered in neutron transport and radiative transfer theories," *J. Math. Phys.* **11**(2), 568–577 (1970).
19. R. D. M. Garcia and C. E. Siewert, "On the dispersion function in particle transport theory," *Zeitschrift Für Angewandte Mathematik Und Physik Zamp* **33**(6), 801–806 (1994).
20. D. Zwillinger and V. Moll, *Table of Integrals, Series, and Products (Eighth Edition)*, (Academic Press, 2015).
21. Z. X. Wang and D. R. Guo, *Special Functions*, (World Scientific, 1989).
22. S. J. Zhang and J. M. Jin, *Computation of Special Functions* (John Wiley & Sons Inc., 1996).
23. N. I. Muskhelishvili, *Singular Integral Equations: Boundary Problems of Function Theory and Their Application to Mathematical Physics* (Dover Publications, 2011).
24. E. M. Stein and R. Shakarchi, *Complex Analysis* (Princeton University Press, 2003).
25. K. M. Case, "Scattering theory, orthogonal polynomials, and the transport equation," *J. Math. Phys.* **15**(7), 974–983 (1974).
26. C. E. Siewert and J. R. T. Jr, "A particular solution for the P_N method in spherical geometry," *J. Quant. Spectrosc. Radiat. Transf.* **46**(5), 459–462 (1991).
27. R. D. M. Garcia, "A P_N particular solution for the radiative transfer equation in spherical geometry," *J. Quant. Spectrosc. Radiat. Transf.* **196**, 155–158 (2017).

1. Introduction

In our first accompanying paper we have derived Case's singular eigenfunctions (CSEs) solution of the light transport equation in the infinite homogeneous tissue with m-dependent anisotropic scattering kernel and benchmarked four computational methods for the m-dependent Chandrasekhar orthogonal polynomials. In this paper we presented the corresponding Fourier transform solution and then showed the consistence of the solutions obtained from the two approaches.

The CSEs solutions are always seen as the exact solutions of integro-differential transport equations and naturally suitable for the light transport in turbid tissue [1–9]. Fourier transform method is a useful method and applied successfully to the transport equations in the regular space [10–17]. Ganapol [12–16] derived the Fourier transform solution for the infinite medium with azimuthal symmetric scattering kernel and showed the consistent theory in a more mathematically unified manner. Machida [17] used the Fourier transform and rotated reference frames technique to compute three-dimension Green's function for three-dimension linear Boltzmann equation. Their outstanding work opened up a new scope to solve the exact radiance distributions that we are seeking for the light transport in biological tissue.

Firstly, by introducing the associated Chandrasekhar polynomials of the second kind, the general analytical solutions for three-term recurrence in Fourier k-domain are extended to the azimuth-dependent or m-dependent case explicitly as the base of the Fourier transform solutions. Secondly, in m-dependent case, taking the associated Chandrasekhar polynomials of the first/second kind as the Legendre moments, the generalized singular eigenfunctions of the first/second kind (GSE1, 2) are defined, in the limit version and in the truncated version, in terms of the associated Legendre polynomials of the first kind with the same degrees and orders. And also, the implicit corresponding relations between GSEs and CSEs are pointed out and proven from the perspective of both expression forms and intrinsic behaviors. Depending on the first Legendre moment, the general analytical solution can be expressed completely and used to Fourier inversion. Lastly, in the Fourier inversion, the contributions from the poles and the branch cut are combined to the final Fourier transform solution. The poles in complex plane correspond to the discrete values in Case's method and the branch cut corresponds to the continuous spectrum, and then the final solution confirms perfectly to the CSEs solution.

During the derivation, many identities are different from those in the case $m = 0$, such as the Christoffel-Darboux (C-D) identities or Liouville-Ostrogradski (L-O) formulas [14–19], Λ_L^m etc. The derivations of these identities are lengthy but very helpful to future study.

2. M-dependent polynomials and C-D formulas

In our following derivations, many m-dependent polynomials or functions and corresponding C-D formulas will be used frequently. However, there are some aspects different from that of the case $m = 0$. In this section, we give brief reviews of these polynomials and formulas.

In the real domain [20–22], $P_l(\mu)$ ($-1 \leq \mu \leq 1$) is the Legendre polynomial of the first kind of degree l . $P_l^m(\mu)$ ($-1 \leq \mu \leq 1, l \geq m \geq 0$) is the associated Legendre polynomial of the first kind of order m and degree l , and the Hobson's definition is

$$P_l^m(\mu) = (-1)^m (1 - \mu^2)^{m/2} \frac{d^m P_l(\mu)}{d\mu^m} \quad (1)$$

$P_l^m(\mu)$ satisfies the three-term recurrence relation

$$\mu(2l+1)P_l^m(\mu) - (l-m+1)P_{l+1}^m(\mu) - (l+m)P_{l-1}^m(\mu) = 0 \quad (2)$$

In the complex plane, $P_l(z)$ is a Legendre orthogonal polynomial [20] and $P_l^m(z)$ is defined as [20–22]

$$P_l^m(z) = (z^2 - 1)^{m/2} \frac{d^m P_l(z)}{dz^m} \quad (3)$$

and satisfies the Eq. (2) with initial terms [17]

$$P_m^m(z) = (2m-1)!! (z^2 - 1)^{m/2}, \quad P_{m+1}^m(z) = (2m+1)zP_m^m(z) \quad (4)$$

Because $P_l(z)$ and $P_l^m(z)$ are polynomials in z , they are analytic in the complex plane.

$Q_l(z)$ is the Legendre function of the second kind and defined in the complex plane excluding a cut line $-\infty$ to 1. However, in our research we usually use the following definition

$$Q_l(z) = \frac{1}{2} \int_{-1}^1 \frac{P_l(v)}{z-v} dv \quad (5)$$

where z is not on the cut line -1 to 1. Similarly, we define the associated Legendre function of the second kind as [17, 20]

$$Q_l^m(z) = \frac{(-1)^m}{2(z^2 - 1)^{m/2}} \int_{-1}^1 \frac{(1 - \mu^2)^{m/2} P_l^m(\mu)}{z - \mu} d\mu \quad (6)$$

where z is in the complex plane with a cut along the interval $(-1, 1)$ on the real axis. From Eq. (6), we can get the three-term recurrence for $Q_l^m(z)$

$$z(2l+1)Q_l^m(z) - (l-m+1)Q_{l+1}^m(z) - (l+m)Q_{l-1}^m(z) = \frac{(2m)!}{(2m-1)!!} (z^2 - 1)^{-m/2} \delta_{lm} \quad (7)$$

with initial terms [17]

$$Q_m^m(z) = \frac{[(2m-1)!!]^2}{2P_m^m(z)} \int_{-1}^1 \frac{(1-\mu^2)^m}{z-\mu} d\mu, \quad Q_{m+1}^m(z) = (2m+1)zQ_m^m(z) - \frac{(2m)!}{P_m^m(z)} \quad (8)$$

where δ_{lm} is the Kronecker delta, $\delta_{lm} = 1$ for $l = m$, and otherwise $\delta_{lm} = 0$.

The Chandrasekhar polynomials $g_l^m(z)$ and $\rho_l^m(z)$ are defined in our first accompanying paper and can be extended into the complex plane. They satisfy the three-term recurrence relations

$$zh_l g_l^m(z) - (l-m+1)g_{l+1}^m(z) - (l+m)g_{l-1}^m(z) = 0 \quad (9)$$

$$zh_l \rho_l^m(z) - (l-m+1)\rho_{l+1}^m(z) - (l+m)\rho_{l-1}^m(z) = -\rho_{m+1}^m(z)\delta_{lm} \quad (10)$$

where $h_l = (2l+1)(1-\varpi f_l)$ and $0 \leq m \leq l$. The initial terms are

$$g_m^m(z) = \frac{(2m)!}{2^m m!} = (2m-1)!!, \quad g_{m+1}^m = zh_m g_m^m(z) \quad (11)$$

$$\rho_m^m(z) = 0, \quad \rho_{m+1}^m = \frac{z}{(2m-1)!!} \quad (12)$$

The Christoffel-Darboux (C-D) identities or Liouville-Ostrogradski (L-O) formulas about above polynomials are ordinary and not difficult to derive, however, in our case $m > 0$, there are still some points that deserve careful consideration. For example, the definitions and three-term recurrence of $Q_l^m(z)$ should be noticed during the derivation. We have already derived these formulas, and listed here for future use. In the following formulas, L is the truncated position of the phase function, that is to say when $l > L$, $f_l = 0$. N is an integer, and $N > L$. $\varpi = \mu_s/\mu_t$ is the single particle albedo, where μ_s is the scattering coefficient, μ_a is the absorption coefficient, and $\mu_t = \mu_a + \mu_s$ is the total attenuation coefficient.

$$\sum_{l=m}^N \frac{(l-m)!}{(l+m)!} (2l+1) g_l^m(z) P_l^m(\mu) = \frac{\varpi z}{z-\mu} \sum_{l=m}^N \frac{(l-m)!}{(l+m)!} (2l+1) f_l g_l^m(z) P_l^m(\mu) + \frac{(N+1-m)!}{(N+m)!} \frac{g_{N+1}^m(z) P_N^m(\mu) - g_N^m(z) P_{N+1}^m(\mu)}{z-\mu} \quad (13)$$

$$\sum_{l=m}^N \frac{(l-m)!}{(l+m)!} (2l+1) \rho_l^m(z) P_l^m(\mu) = \frac{\varpi z}{z-\mu} \sum_{l=m}^N \frac{(l-m)!}{(l+m)!} (2l+1) f_l \rho_l^m(z) P_l^m(\mu) + \frac{(N+1-m)!}{(N+m)!} \frac{\rho_{N+1}^m(z) P_N^m(\mu) - \rho_N^m(z) P_{N+1}^m(\mu)}{z-\mu} - \frac{z}{z-\mu} \frac{(1-\mu^2)^{m/2}}{(2m)!} \quad (14)$$

$$\varpi z \sum_{l=m}^L \frac{(l-m)!}{(l+m)!} (2l+1) f_l g_l^m(z) P_l^m(z) = \frac{(L+1-m)!}{(L+m)!} [g_L^m(z) P_{L+1}^m(z) - g_{L+1}^m(z) P_L^m(z)] \quad (15)$$

$$\varpi z \sum_{l=m}^L \frac{(l-m)!}{(l+m)!} (2l+1) f_l g_l^m(z) Q_l^m(z) = \frac{(L+1-m)!}{(L+m)!} [g_L^m(z) Q_{L+1}^m(z) - g_{L+1}^m(z) Q_L^m(z)] + (z^2 - 1)^{-m/2} \quad (16)$$

$$\varpi z \sum_{l=m}^L \frac{(l-m)!}{(l+m)!} (2l+1) f_l \rho_l^m(z) P_l^m(z) = \frac{(L+1-m)!}{(L+m)!} [\rho_L^m(z) P_{L+1}^m(z) - \rho_{L+1}^m(z) P_L^m(z)] + \frac{z(1-z^2)^{m/2}}{(2m)!} \quad (17)$$

$$\varpi z \sum_{l=m}^L \frac{(l-m)!}{(l+m)!} (2l+1) f_l \rho_l^m(z) Q_l^m(z) = \frac{(L+1-m)!}{(L+m)!} [\rho_L^m(z) Q_{L+1}^m(z) - \rho_{L+1}^m(z) Q_L^m(z)] + \frac{\rho_{m+1}^m(z) Q_m^m(z)}{(2m)!} \quad (18)$$

$$\sum_{l=m}^N \frac{(l-m)!}{(l+m)!} \frac{2l+1}{2} P_l^m(z) P_l^m(\mu) = \frac{1}{2} \frac{(N+1-m)!}{(N+m)!} \frac{P_{N+1}^m(z) P_N^m(\mu) - P_N^m(z) P_{N+1}^m(\mu)}{z - \mu} \quad (19)$$

$$\sum_{l=m}^N \frac{(l-m)!}{(l+m)!} \frac{2l+1}{2} Q_l^m(z) P_l^m(\mu) = \frac{1}{2} \frac{(N+1-m)!}{(N+m)!} \frac{Q_{N+1}^m(z) P_N^m(\mu) - Q_N^m(z) P_{N+1}^m(\mu)}{z - \mu} + \frac{1}{2} \frac{(1-\mu^2)^{m/2} (z^2 - 1)^{-m/2}}{z - \mu} \quad (20)$$

$$g_{l-1}^m(z) \rho_l^m(z) - g_l^m(z) \rho_{l-1}^m(z) = \frac{(l+m-1)!}{(l-m)!} \frac{z}{(2m)!} \quad (21)$$

$$P_{l+1}^m(z) Q_l^m(z) - P_l^m(z) Q_{l+1}^m(z) = \frac{(l+m)!}{(l+1-m)!} \quad (22)$$

3. Standard Fourier transform and matrix inversion

Let's begin with the Eq. (4) in our first accompanying paper:

$$\mu \frac{\partial \psi^m(\tau, \mu; \mu_0)}{\partial \tau} + \psi^m(\tau, \mu; \mu_0) = \frac{\varpi}{2} \sum_{l=|m|}^L \beta_l^m P_l^m(\mu) \int_{-1}^{+1} P_l^m(\mu') \psi^m(\tau, \mu'; \mu_0) d\mu' + \delta(\tau) \delta(\mu - \mu_0) e^{-im\phi_0} \quad (23)$$

where $\tau = \mu_l x$ is the optical distance which changes the absolute position variable x into optical depth τ , and $\beta_l^m = \frac{(l-m)!}{(l+m)!} (2l+1) f_l$. Equation (23) describes the distribution of the

radiances in the infinite homogeneous tissue for specific m . Because of the spherical symmetric attribute of the scattering, in the following derivation we assume that $m \geq 0$ and $l \geq m$.

Then, we introduce the Fourier transform and inversion

$$\tilde{\psi}^m(k, \mu; \mu_0) = \int_{-\infty}^{\infty} \psi^m(\tau, \mu; \mu_0) e^{-ik\tau} d\tau \quad (24)$$

$$\psi^m(\tau, \mu; \mu_0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{\psi}^m(k, \mu; \mu_0) e^{ik\tau} dk \quad (25)$$

and apply Eq. (24) to Eq. (23), and we get

$$(1 + ik\mu) \tilde{\psi}^m(k, \mu; \mu_0) = \frac{\varpi}{2} \sum_{l=m}^L \beta_l^m P_l^m(\mu) \int_{-1}^{+1} P_l^m(\mu') \tilde{\psi}^m(k, \mu'; \mu_0) d\mu' + \delta(\mu - \mu_0) e^{-im\varphi_0} \quad (26)$$

Expanding the $\tilde{\psi}^m(k, \mu; \mu_0)$ by the associated Legendre polynomials of the first kind

$$\tilde{\psi}^m(k, \mu; \mu_0) = \sum_{l=m}^{\infty} \frac{(l-m)!}{(l+m)!} \frac{2l+1}{2} \tilde{\psi}_l^m(k; \mu_0) P_l^m(\mu) \quad (27)$$

and letting $z = i/k$, and we get the Legendre moments

$$\tilde{\psi}_j^m(k; \mu_0) = \varpi \sum_{l=m}^L \beta_l^m L_{j,l}^m(z) \tilde{\psi}_l^m(k; \mu_0) + \frac{z}{z - \mu_0} P_j^m(\mu_0) e^{-im\varphi_0} \quad (28)$$

where $j = m, m+1, \dots$, and

$$L_{j,l}^m(z) = \frac{z}{2} \int_{-1}^1 \frac{P_j^m(\mu) P_l^m(\mu)}{z - \mu} d\mu = \begin{cases} z P_l^m(z) Q_j^m(z), & l \leq j \\ z Q_l^m(z) P_j^m(z), & l > j \end{cases} \quad (29)$$

In the derivation from Eq. (27) to Eq. (29), k is constrained in the complex plane excluding cut lines $[i, \infty)$ and $[-i, -\infty)$ on the imaginary axis, and correspondingly z is defined in the complex plane excluding a cut line -1 to $+1$ on the real axis.

Due to $\beta_l^m = 0$ for $l > L$, when $m \leq j \leq L$, Eq. (28) can be rewritten as

$$\sum_{l=m}^L [\delta_{jl} - \varpi \beta_l^m L_{j,l}^m(z)] \tilde{\psi}_{L,l}^m(k; \mu_0) = \frac{z}{z - \mu_0} P_j^m(\mu_0) e^{-im\varphi_0} \quad (30)$$

where δ_{jl} is the Kronecker delta, $\delta_{jl} = 1$ for $j = l$, and otherwise $\delta_{jl} = 0$. The subscript L in $\tilde{\psi}_{L,l}^m(k, \mu; \mu_0)$ denotes the truncation. We define the following matrices and vectors

$$\begin{bmatrix} \tilde{\psi}^m \end{bmatrix}_l = \tilde{\psi}_{L,l}^m(k; \mu_0) \quad (31)$$

$$\begin{bmatrix} L^m \end{bmatrix}_{j,l} = L_{j,l}^m(z) \quad (32)$$

$$\begin{bmatrix} \beta^m \end{bmatrix}_{j,l} = \beta_l^m \delta_{jl} \quad (33)$$

$$\left[\mathbf{P}^m \right]_l = P_l^m(\mu_0) \quad (34)$$

and Eq. (30) can be rewritten as

$$\left[\mathbf{I} - \varpi \mathbf{L}^m \boldsymbol{\beta}^m \right] \tilde{\boldsymbol{\psi}}^m = \frac{z}{z - \mu_0} e^{-im\varphi_0} \mathbf{P}^m \quad (35)$$

Matrix inversion can be directly used to get the moment vector of $\tilde{\boldsymbol{\psi}}_{L,l}^m(k, \mu; \mu_0)$

$$\tilde{\boldsymbol{\psi}}^m = \frac{z}{z - \mu_0} e^{-im\varphi_0} \left[\mathbf{I} - \varpi \mathbf{L}^m \boldsymbol{\beta}^m \right]^{-1} \mathbf{P}^m \quad (36)$$

Then, from Eq. (25), Fourier transform inversion can be applied to Eq. (36) to get the final conventional solution, and we don't deduce further.

4. Fourier domain solution and GSEs

4.1 Solution of three-term recurrence

From Eq. (4), we can also get the three-term recurrence

$$zh_l \tilde{\boldsymbol{\psi}}_l^m(k; \mu_0) - (l - m + 1) \tilde{\boldsymbol{\psi}}_{l+1}^m(k; \mu_0) - (l + m) \tilde{\boldsymbol{\psi}}_{l-1}^m(k; \mu_0) = z S_l^m(\mu_0) \quad (37)$$

where $S_l^m(\mu_0) = (2l + 1) P_l^m(\mu_0) e^{-im\varphi_0}$. As Inönü pointed out [18], the finite-difference equation, the homogeneous equation of Eq. (37), has two linear independent polynomial-type solutions, e.g. $g_l^m(z)$ and $\rho_l^m(z)$, which are the associated Chandrasekhar polynomials of the first and second kind respectively and defined in our first accompanying paper. Ganapol [14] defined the solution form of the three-term recurrence as summation of the general solution of the homogeneous equation and the particular solution of the inhomogeneous equation, and derived the form of the particular solution. The solution form is also applied by Ganapol to derive the Fourier transform solution in case $m = 0$ [12–16] and by Machida to express the solution in m-dependent case [17]. Here, we still use the solution form and define

$$\tilde{\boldsymbol{\psi}}_l^m(k; \mu_0) = a^m(z; \mu_0) g_l^m(z) + b^m(z; \mu_0) \rho_l^m(z) + z \sum_{j=m}^l \eta_{l,j}^m(z) S_j(\mu_0) \quad (38)$$

where $a^m(z; \mu_0)$, $b^m(z; \mu_0)$ and $\eta_j^m(z)$ are the coefficients to be determined. It is noted that the analytical solution form expressed in Eq. (38) may be different from that given by Machida [17], but they are identical essentially.

We assume that the first Legendre moment of $\tilde{\boldsymbol{\psi}}^m(k; \mu_0)$, $\tilde{\boldsymbol{\psi}}_m^m(k; \mu_0)$, is known, then we let $l = m$, $a^m(z; \mu_0) = \tilde{\boldsymbol{\psi}}_m^m(k; \mu_0) / g_m^m(z)$, and $b^m(z; \mu_0) = 0$ in Eq. (38) and get

$$\tilde{\boldsymbol{\psi}}_l^m(k; \mu_0) = \frac{\tilde{\boldsymbol{\psi}}_m^m(k; \mu_0)}{g_m^m(z)} g_l^m(z) + z \sum_{j=m}^l \eta_{l,j}^m(z) S_j(\mu_0) \quad (39)$$

Substituting Eq. (39) into Eq. (37), to keep balance between the equations, the following relations must be satisfied

$$zh_l \eta_{l,j}^m(z) - (l - m + 1) \eta_{l+1,j}^m(z) - (l + m) \eta_{l-1,j}^m(z) = 0 \quad (40)$$

$$\eta_{l,l}^m = 0 \quad (41)$$

$$\eta_{l+1,l}^m(z) = -1/(l-m+1) \quad (42)$$

Equation (40) is a three-term recurrence, too. Considering that Eq. (40) is a homogeneous equation, the particular solution is zero, and $\eta_{l,j}^m(z)$ can be expressed by general solution form

$$\eta_{l,j}^m(z) = c_j^m(z) g_l^m(z) + d_j^m(z) \rho_l^m(z) \quad (43)$$

The coefficients $c_j^m(z), d_j^m(z)$ are determined by substituting Eq. (43) into Eq. (41) and Eq. (42)

$$c_j^m(z) = \frac{\rho_j^m(z)}{j-m+1} \frac{1}{\rho_j^m(z) g_{j+1}^m(z) - g_j^m(z) \rho_{j+1}^m(z)} \quad (44)$$

$$d_j^m(z) = \frac{g_j^m(z)}{j-m+1} \frac{1}{g_j^m(z) \rho_{j+1}^m(z) - \rho_j^m(z) g_{j+1}^m(z)} \quad (45)$$

From Eq. (21), $\eta_{l,j}^m(z)$ now is

$$\eta_{l,j}^m(z) = \frac{(j-m)! (2m)!}{(j+m)! z} [\rho_j^m(z) g_l^m(z) - g_j^m(z) \rho_l^m(z)] \quad (47)$$

Thus, the analytical solution Eq. (37) can be expressed as

$$\tilde{\psi}_l^m(k; \mu_0) = \frac{\tilde{\psi}_m^m(k; \mu_0)}{g_m^m(z)} g_l^m(z) + \chi_l^m(z, \mu_0) \quad (48)$$

where

$$\chi_l^m(z, \mu) \equiv (2m)! \sum_{j=m}^l \frac{(j-m)!}{(j+m)!} [\rho_j^m(z) g_l^m(z) - g_j^m(z) \rho_l^m(z)] S_j(\mu) \quad (49)$$

$\chi_l^m(z, \mu)$ is the auxiliary function and $\chi_m^m(z, \mu) = 0$. From Eq. (48) and Eq. (49), we can see that if the first Legendre moment is known, the other Legendre moments will be easily got. The first Legendre moment plays an important part during the derivation.

4.2 The first Legendre moment

The first Legendre moment is so important that Ganapol had provided several methods of solving it [12–16]. When $m=0$, those methods are all valid. However, when $m>0$, it is impossible to use the isotropic point source to find the first Legendre moment. So, we substitute the Eq. (48) into Eq. (30), let $j=m$, and use

$$L_{m,l}^m(z) = \frac{z}{2} \int_{-1}^1 \frac{P_m^m(\mu) P_l^m(\mu)}{z-\mu} d\mu = z P_m^m(z) Q_l^m(z) \quad (50)$$

then we get

$$\tilde{\psi}_m^m(k; \mu_0) = \frac{1}{\Lambda_m^m(z)} \left[\frac{z}{z-\mu_0} P_m^m(\mu_0) e^{-im\varphi_0} + \varpi z P_m^m(z) \sum_{l=m}^L \beta_l^m Q_l^m(z) \chi_l^m(z, \mu_0) \right] \quad (51)$$

where

$$\Lambda^m(z) \equiv 1 - \varpi z \frac{P_m^m(z)}{g_m^m(z)} \sum_{l=m}^L \beta_l^m g_l^m(z) Q_l^m(z) \quad (52)$$

In our first accompanying paper, we defined a holomorphic function in the complex plane excluding a cut line -1 to 1 on the real axis

$$\Lambda^m(z) \equiv 1 - \frac{\varpi z}{2} \int_{-1}^1 \frac{g(z, \mu)}{z - \mu} (1 - \mu^2)^{m/2} d\mu \quad (53)$$

where

$$g^m(z, \mu) \equiv \sum_{l=m}^L \beta_l^m g_l^m(z) P_l^m(\mu) \quad (54)$$

Substituting $P_m^m(z) = (2m-1)!!(z^2-1)^{m/2}$ and $g_m^m(z) = (2m-1)!!$ into Eq. (52), it is obvious that Eq. (52) and Eq. (53) are identical.

4.3 The generalized singular eigenfunctions of the first kind

Now, we can begin to build the generalized singular eigenfunctions of the first kind (GSE1) which are extended from the Ganapol's definition for the case $m=0$ in the complex z -plane [15, 16].

Let's define GSE1, called the limit version, as the expansion series of the combination of the associated Chandrasekhar polynomials of the first kind and the associated Legendre polynomials of first kind

$$\phi_L^m(z, \mu) \equiv \sum_{l=m}^{\infty} \frac{(l-m)!}{(l+m)!} \frac{2l+1}{2} g_l^m(z) P_l^m(\mu) \quad (55)$$

and define the corresponding truncated version

$$\phi_L^m(z, \mu; N) \equiv \sum_{l=m}^N \frac{(l-m)!}{(l+m)!} \frac{2l+1}{2} g_l^m(z) P_l^m(\mu) \quad (56)$$

where $N > L$. As $N \rightarrow \infty$, $\phi_L^m(z, \mu; N) \rightarrow \phi_L^m(z, \mu)$.

Substituting Eq. (13) into Eq. (56), we get

$$\phi_L^m(z, \mu; N) = \frac{\varpi z}{2} \frac{\varphi_N^m(z, \mu)}{z - \mu} + \frac{1}{2} \frac{(N-m+1)!}{(N+m)!} \frac{g_{N+1}^m(z) P_N^m(\mu) - g_N^m(z) P_{N+1}^m(\mu)}{z - \mu} \quad (57)$$

where

$$\varphi_N^m(z, \mu) \equiv \sum_{l=m}^N \frac{(l-m)!}{(l+m)!} (2l+1) f_l g_l^m(z) P_l^m(\mu) \quad (58)$$

Because $l > L$, $f_l = 0$, $\varphi_N^m(z, \mu)$ will remain unchanged for any $N > L$,

$$\varphi_N^m(z, \mu) = \varphi_L^m(z, \mu) = \sum_{l=m}^L \frac{(l-m)!}{(l+m)!} (2l+1) f_l g_l^m(z) P_l^m(\mu) \quad (59)$$

From Eq. (56), the analyticity of $\phi_L^m(z, \mu; N)$ depends on that of $g_l^m(z)$. When $l \leq L$, as discussed in our first accompanying paper, it is certain that $g_l^m(z)$ is an analytic function because it can be expressed with polynomials explicitly or a tri-diagonal determinant which

can be factorized with the eigenvalues. When $l > L$, from Eq. (9), the three-term recurrence for $g_l^m(z)$ will be the homogeneous form

$$z(2l+1)g_l^m(z) - (l-m+1)g_{l+1}^m(z) - (l+m)g_{l-1}^m(z) = 0 \quad (60)$$

Considering the $g_l^m(z)$ degenerates into associated Legendre polynomials when $l > L$, we can use the form of the general analytical solution of three-term recurrence again

$$g_l^m(z) = \alpha(z)P_l^m(z) + \beta(z)Q_l^m(z) \quad (61)$$

According to the continuity of $g_l^m(z)$ at the truncated position, we have

$$\begin{cases} g_L^m(z) = \alpha(z)P_L^m(z) + \beta(z)Q_L^m(z) \\ g_{L+1}^m(z) = \alpha(z)P_{L+1}^m(z) + \beta(z)Q_{L+1}^m(z) \end{cases} \quad (62)$$

Using Eq. (22), the coefficients can be solved

$$\begin{cases} \alpha(z) = \Lambda_L^m(z) = \frac{(L-m+1)!}{(L+m)!} [g_{L+1}^m(z)Q_L^m(z) - g_L^m(z)Q_{L+1}^m(z)] \\ \beta(z) = \psi_L^m(z) = -\frac{(L-m+1)!}{(L+m)!} [g_{L+1}^m(z)P_L^m(z) - g_L^m(z)P_{L+1}^m(z)] \end{cases} \quad (63)$$

So when $l > L$,

$$g_l^m(z) = \Lambda_L^m(z)P_l^m(z) + \psi_L^m(z)Q_l^m(z) \quad (64)$$

or

$$g_l^m(z) = \frac{(L-m+1)!}{(L+m)!} \left\{ g_L^m(z) [P_{L+1}^m(z)Q_l^m(z) - Q_{L+1}^m(z)P_l^m(z)] \right. \\ \left. + g_{L+1}^m(z) [P_l^m(z)Q_L^m(z) - P_L^m(z)Q_l^m(z)] \right\} \quad (65)$$

By letting $l = L$ or $l = L+1$ in Eq. (65), the correctness of the expression of $g_l^m(z)$ is apparent. However, due to the singularity of $Q_l^m(z)$, the analyticity of $g_l^m(z)$ needs to be proven. When $m = 0$, Ganapol [15, 16] proved it by the relationship between $Q_l(z)$ and $P_l(z)$. In fact, from the definition of $Q_l^m(z)$ in Eq. (6), we write down the part of Eq. (65)

$$\begin{aligned} & P_l^m(z)Q_L^m(z) - P_L^m(z)Q_l^m(z) \\ &= \frac{(-1)^m}{2(2m-1)!!} \int_{-1}^1 \left[\frac{d^m P_l(z)}{dz^m} P_L^m(\mu) - \frac{d^m P_L(z)}{dz^m} P_l^m(\mu) \right] \frac{P_m^m(\mu)}{z - \mu} d\mu \end{aligned} \quad (66)$$

Because $P_l(z)$ is a polynomial in z of degree l , and $P_l^m(\mu)$ is a polynomial in μ of degree l , it can predicate that the denominator will vanish and the expression is analytic. So we can conclude that $g_l^m(z)$ is an analytic polynomials in the complex z -plane and then $\psi_L^m(z)$ is analytic, too. So, $\phi_L^m(z, \mu; N)$ is actually analytic function in the complex z -plane. But from Eq. (63), $\Lambda_L^m(z)$ is only analytic excluding a cut line -1 to 1 on the real axis.

Substituting Eq. (64) in Eq. (56), we get the truncated version of GSE1

$$\phi_L^m(z, \mu; N) = \frac{\sigma z}{2} \frac{\phi_L^m(z, \mu)}{z - \mu} + \Lambda_L^m(z) \delta_N^m(z, \mu) + \psi_L^m(z) q_N^m(z, \mu) \quad (67)$$

where

$$\delta_N^m(z, \mu) = \frac{1}{2} \frac{(N+1-m)!}{(N+m)!} \frac{P_{N+1}^m(z) P_N^m(\mu) - P_N^m(z) P_{N+1}^m(\mu)}{z - \mu} \quad (68)$$

$$q_N^m(z, \mu) = \frac{1}{2} \frac{(N+1-m)!}{(N+m)!} \frac{Q_{N+1}^m(z) P_N^m(\mu) - Q_N^m(z) P_{N+1}^m(\mu)}{z - \mu} \quad (69)$$

From Eq. (54) and Eq. (59), $\phi_L^m(z, \mu)$ is identical to $g^m(z, \mu)$ in Case's method. Thus, the truncated version of GSE1, $\phi_L^m(z, \mu; N)$, is expressed by all known functions and similar to the eigenfunctions in Case's method.

Now we can inspect the relations between the CSEs and GSEs. Let's check the discrete eigenfunctions first. Because the discrete eigenvalues in Case's method are real and their absolute values are greater than 1, here we let z is real number, i.e. v , and $|v| > 1$. Though $\delta(v - \mu)$ will vanish for discrete eigenvalues, expanding the Dirac delta function in terms of the associated Legendre polynomials of the first kind, and we get

$$\delta(v - \mu) = \sum_{l=m}^{\infty} \frac{(l-m)!}{(l+m)!} \frac{2l+1}{2} P_l^m(v) P_l^m(\mu) \quad (70)$$

From Eq. (19), when $z = v$ we get

$$\delta_N^m(v, \mu) = \sum_{l=m}^N \frac{(l-m)!}{(l+m)!} \frac{2l+1}{2} P_l^m(v) P_l^m(\mu) \quad (71)$$

So

$$\lim_{N \rightarrow \infty} \delta_N^m(v, \mu) = \delta(v - \mu) \quad (72)$$

It should be noticed that Eq. (70) ~ Eq. (72) are only valid in real domain. In complex domain, Ganapol suggested [16] that the Dirac delta function should be viewed as a placeholder and the limit came at the end when z was taken to the cut line to become real.

Expanding $\frac{1}{2} \frac{(1-\mu^2)^{m/2} (v^2-1)^{-m/2}}{v-\mu}$ in terms of the associated Legendre polynomials of the first kind, we get

$$\frac{1}{2} \frac{(1-\mu^2)^{m/2} (v^2-1)^{-m/2}}{v-\mu} = \sum_{l=m}^{\infty} \frac{(l-m)!}{(l+m)!} \frac{2l+1}{2} Q_l^m(v) P_l^m(\mu) \quad (73)$$

From Eq. (20), we get

$$q_N^m(v, \mu) = \sum_{l=m}^N \frac{(l-m)!}{(l+m)!} \frac{2l+1}{2} Q_l^m(v) P_l^m(\mu) - \frac{1}{2} \frac{(1-\mu^2)^{m/2} (v^2-1)^{-m/2}}{v-\mu} \quad (74)$$

So

$$\lim_{N \rightarrow \infty} q_N^m(v, \mu) = 0 \quad (75)$$

Then, as $N \rightarrow \infty$ and v is real discrete eigenvalues, the limit version of GSE1 is

$$\phi_L^m(v, \mu) = \lim_{N \rightarrow \infty} \phi_L^m(v, \mu) = \frac{\varpi v \phi_L^m(v, \mu)}{2(v - \mu)} \quad (76)$$

It is obvious that Eq. (76) is consistent with the discrete eigenfunctions Eq. (8) in our first accompanying paper. However, for the continuous spectrum in Case's method, it can't immediately conclude that $\phi_L^m(z, \mu; N)$ is identical to the continuous eigenfunctions in Case's method. To prove that fact, we need to prove the consistency from both expression forms and intrinsic behaviors. So, firstly, we discuss how $\phi_L^m(z, \mu; N)$ approaches the cut line -1 to 1 from above (+) and below (-). Because of the analyticity of $\phi_L^m(z, \mu)$ and $\psi_L^m(z)$ in the complex z -plane, when $v \in [-1, 1]$ we define

$$\begin{aligned} [\phi_L^m(v, \mu; N)]^\pm &= \frac{\varpi v}{2} \phi_L^m(v, \mu) \left[\frac{1}{z - \mu} \right]^\pm \\ &+ [\Lambda_L^m(v)]^\pm [\delta_N^m(v, \mu)]^\pm + \psi_L^m(v) [q_N^m(v, \mu)]^\pm \end{aligned} \quad (77)$$

From Plemelj's formula [23] we get

$$\left[\frac{1}{z - \mu} \right]^\pm = \mathcal{P} \frac{1}{v - \mu} \mp i\pi \delta(v - \mu) \quad (78)$$

Here \mathcal{P} denotes the Cauchy principal value. From Eq. (6) and Plemelj's formula we get

$$[Q_L^m(v)]^\pm = \mathcal{P} Q_L^m(v) \mp \frac{i\pi}{2} P_L^m(v) \quad (79)$$

So, from Eq. (63)

$$\begin{aligned} [\Lambda_L^m(v)]^\pm &= \frac{(L - m + 1)!}{(L + m)!} \left\{ g_{L+1}^m(v) [Q_L^m(v)]^\pm - g_L^m(v) [Q_{L+1}^m(v)]^\pm \right\} \\ &= \mathcal{P} \Lambda_L^m(v) \pm \frac{i\pi}{2} \psi_L^m(v) \end{aligned} \quad (80)$$

From Eq. (68) we get

$$\begin{aligned} [\delta_N^m(v, \mu)]^\pm &= \frac{1}{2} \frac{(N + 1 - m)!}{(N + m)!} \left[P_{N+1}^m(v) P_N^m(\mu) - P_N^m(v) P_{N+1}^m(\mu) \right] \left[\frac{1}{z - \mu} \right]^\pm \\ &= \mathcal{P} \delta_N^m(v, \mu) \end{aligned} \quad (81)$$

From Eq. (69) we get

$$\begin{aligned} [q_N^m(v, \mu)]^\pm &= \frac{1}{2} \frac{(N + 1 - m)!}{(N + m)!} \left\{ [Q_{N+1}^m(v)]^\pm P_N^m(\mu) - [Q_N^m(v)]^\pm P_{N+1}^m(\mu) \right\} \left[\frac{1}{z - \mu} \right]^\pm \\ &= \mathcal{P} q_N^m(v, \mu) \mp \frac{i\pi}{2} [\mathcal{P} \delta_N^m(v, \mu) - \delta(v - \mu)] \end{aligned} \quad (82)$$

Substituting Eq. (78), Eq. (80) ~Eq. (82) into Eq. (77) we get

$$\begin{aligned}
\left[\phi_L^m(v, \mu; N)\right]^\pm &= \frac{\varpi v}{2} \phi_L^m(v, \mu) \mathcal{P} \frac{1}{v-\mu} + \mathcal{P} \Lambda_L^m(v) \mathcal{P} \delta_N^m(v, \mu) + \psi_L^m(v) \mathcal{P} q_N^m(v, \mu) \\
&\mp i\pi \frac{\varpi v}{2} \phi_L^m(v, v) \delta(v-\mu) \pm \frac{i\pi}{2} \psi_L^m(v) \mathcal{P} \delta_N^m(v, \mu) \\
&\mp \frac{i\pi}{2} \psi_L^m(v) \left[\mathcal{P} \delta_N^m(v, \mu) - \delta(v-\mu)\right]
\end{aligned} \tag{83}$$

From Eq. (13) and Eq. (63), let $\mu = z = v$ and we get

$$\varpi v \phi_L^m(v, v) = \psi_L^m(v) \tag{84}$$

and

$$\left[\phi_L^m(v, \mu; N)\right]^\pm = \frac{\varpi v}{2} \phi_L^m(v, \mu) \mathcal{P} \frac{1}{v-\mu} + \mathcal{P} \Lambda_L^m(v) \mathcal{P} \delta_N^m(v, \mu) + \psi_L^m(v) \mathcal{P} q_N^m(v, \mu) \tag{85}$$

Equation (85) is the evidence of the analyticity of $\phi_L^m(z, \mu; N)$. Since now z become a real number, as $N \rightarrow \infty$, we can safely get the equivalent Case's singular eigenfunctions in the continuous spectrum

$$\phi_L^m(v, \mu) = \frac{\varpi v}{2} \phi_L^m(v, \mu) \mathcal{P} \frac{1}{v-\mu} + \mathcal{P} \Lambda_L^m(v) \delta(v-\mu) \tag{86}$$

Thus, the consistency of intrinsic behaviors is shown. Further, $\mathcal{P} \Lambda_L^m(v)$ should be identical to $\lambda(v)/(1-v^2)^{m/2}$ from the expression forms. In fact, from Eq. (14) in our first accompanying paper, we get

$$\begin{aligned}
\lambda(v) &= 1 - \frac{\varpi v}{2} \mathcal{P} \int_{-1}^1 \frac{g^m(v, \mu)}{v-\mu} (1-\mu^2)^{m/2} d\mu \\
&= 1 - \frac{\varpi v}{2} \sum_{l=m}^L \frac{(l-m)!}{(l+m)!} (2l+1) f_l g_l^m(v) \mathcal{P} \int_{-1}^1 \frac{P_l^m(\mu) (1-\mu^2)^{m/2}}{v-\mu} d\mu \\
&= 1 - \varpi v \sum_{l=m}^L \frac{(l-m)!}{(l+m)!} (2l+1) f_l g_l^m(v) \mathcal{P} \left[(1-v^2)^{m/2} Q_l^m(v) \right]
\end{aligned} \tag{87}$$

From Eq. (16) and Eq. (63), we get

$$\Lambda_L^m(z) = (z^2 - 1)^{-m/2} - \varpi z \sum_{l=m}^L \frac{(l-m)!}{(l+m)!} (2l+1) f_l g_l^m(z) Q_l^m(z) \tag{88}$$

We note that when $z \in [-1, 1]$, $(z^2 - 1)^{-m/2}$ should be changed into $(1 - z^2)^{-m/2}$ and $Q_l^m(z)$ should be changed to Cauchy principal value integral. So we proved

$$\mathcal{P} \Lambda_L^m(v) = \frac{\lambda(v)}{(1-v^2)^{m/2}} \tag{89}$$

So the consistency of eigenfunctions form is proven. When $m = 0$, our GSE1 is identical to the Ganapol's GSE1. Of course, GSE1 satisfies the normalization condition Eq. (9) in our first accompanying paper naturally.

4.4 The generalized singular eigenfunction of the second kind

Similar to GSE1, we define the limit version of the generalized singular eigenfunction of the second kind (GSE2)

$$\Theta_L^m(z, \mu) \equiv \sum_{l=m}^{\infty} \frac{(l-m)!}{(l+m)!} \frac{2l+1}{2} \rho_l^m(z) P_l^m(\mu) \quad (90)$$

and the corresponding truncated version

$$\Theta_L^m(z, \mu; N) \equiv \sum_{l=m}^N \frac{(l-m)!}{(l+m)!} \frac{2l+1}{2} \rho_l^m(z) P_l^m(\mu) \quad (91)$$

where $N > L$. As $N \rightarrow \infty$, $\Theta_L^m(z, \mu; N) \rightarrow \Theta_L^m(z, \mu)$. Substituting Eq. (14) into Eq. (91), we get

$$\begin{aligned} \Theta_L^m(z, \mu; N) = & \frac{1}{2} \frac{z}{z-\mu} \left[\varpi h_L^m(z, \mu) - \frac{(1-\mu^2)^{m/2}}{(2m)!} \right] \\ & - \frac{1}{2} \frac{1}{z-\mu} \frac{(N+1-m)!}{(N+m)!} [\rho_N^m(z) P_{N+1}^m(\mu) - \rho_{N+1}^m(z) P_N^m(\mu)] \end{aligned} \quad (92)$$

where

$$h_N^m(z, \mu) \equiv h_L^m(z, \mu) = \sum_{l=m}^L \frac{(l-m)!}{(l+m)!} (2l+1) f_l \rho_l^m(z) P_l^m(\mu) \quad (93)$$

As the derivation in the previous section, when $l > L$ we get

$$\rho_l^m(z) = \gamma_l^m(z) P_l^m(z) + \theta_l^m(z) Q_l^m(z) \quad (94)$$

where

$$\begin{cases} \gamma_L^m(z) = \frac{(L-m+1)!}{(L+m)!} [\rho_{L+1}^m(z) Q_L^m(z) - \rho_L^m(z) Q_{L+1}^m(z)] \\ \theta_L^m(z) = -\frac{(L-m+1)!}{(L+m)!} [\rho_{L+1}^m(z) P_L^m(z) - \rho_L^m(z) P_{L+1}^m(z)] \end{cases} \quad (95)$$

As $g_l^m(z)$ is an analytic function in the complex z -plane, we conclude that $\rho_l^m(z)$ is analytic function without trying to prove it.

Substituting Eq. (94) and Eq. (95) into Eq. (91) we get

$$\Theta_L^m(z, \mu; N) = \frac{1}{2} \frac{z}{z-\mu} \left[\varpi h_L^m(z, \mu) - \frac{(1-\mu^2)^{m/2}}{(2m)!} \right] + \gamma_L^m(z) \delta_N^m(z, \mu) + \theta_L^m(z) q_N^m(z, \mu) \quad (96)$$

Similar to the derivation in previous section, as $N \rightarrow \infty$ and z is discrete eigenvalues, we get

$$\Theta_L^m(v, \mu) = \frac{1}{2} \frac{v}{v-\mu} \left[\varpi h_L^m(v, \mu) - \frac{(1-\mu^2)^{m/2}}{(2m)!} \right] \quad (97)$$

From Eq. (17) and Eq. (97), let $\mu = z = v$ and we get

$$\varpi v h_L^m(v, v) - \frac{v(1-v^2)^{m/2}}{(2m)!} = \theta_L^m(v) \quad (98)$$

Further, when z approaches the cut line -1 to 1 from above (+) and below (-), we can also get

$$\begin{aligned} [\Theta_L^m(v, \mu; N)]^\pm &= \frac{v}{2} \left[\varpi h_L^m(v, \mu) - \frac{(1-\mu^2)^{m/2}}{(2m)!} \right] \mathcal{P} \frac{1}{v-\mu} \\ &\quad + \mathcal{P} \gamma_L^m(v) \mathcal{P} \delta_N^m(v, \mu) + \theta_L^m(v) \mathcal{P} q_N^m(v, \mu) \end{aligned} \quad (99)$$

So as $N \rightarrow \infty$, we get the analogue of the Case's singular eigenfunctions in the continuous spectrum

$$\Theta_L^m(v, \mu) = \frac{v}{2} \left[\varpi h_L^m(v, \mu) - \frac{(1-\mu^2)^{m/2}}{(2m)!} \right] \mathcal{P} \frac{1}{v-\mu} + \mathcal{P} \gamma_L^m(v) \delta(v-\mu) \quad (100)$$

When $m = 0$, our GSE2 is identical to the Ganapol's GSE2 [15, 16].

And also, from Eq. (63) and Eq. (95), we get

$$\Lambda_L^m(z) \theta_L^m(z) - \psi_L^m(z) \gamma_L^m(z) = -\frac{z}{(2m)!} \quad (101)$$

Considering v in Eq. (84) and Eq. (98) can be extended to any z in complex z -plane, we get

$$\Lambda_L^m(z) \left[\varpi h_L^m(z, z) - \frac{(1-z^2)^{m/2}}{(2m)!} \right] - \varpi \varphi_N^m(z, z) \gamma_L^m(z) = -\frac{1}{(2m)!} \quad (102)$$

then

$$\frac{\gamma_L^m(z)}{\Lambda_L^m(z)} = \frac{1}{\varpi \varphi_L^m(z, z)} \left[\frac{1}{(2m)!} \frac{1}{\Lambda_L^m(z)} + \varpi h_L^m(z, z) - \frac{(1-z^2)^{m/2}}{(2m)!} \right] \quad (103)$$

4.5 Fourier domain analytical solution

It is time to build the Fourier domain analytical solution to prepare for the Fourier transform inversion. From Eq. (27) and Eq. (48) we can build the Fourier domain analytical solution

$$\begin{aligned} \tilde{\psi}^m(k, \mu; \mu_0) &= \sum_{l=m}^{\infty} \frac{(l-m)!}{(l+m)!} \frac{2l+1}{2} \tilde{\psi}_l^m(k; \mu_0) P_l^m(\mu) \\ &= \phi_L^m(z, \mu) \frac{\tilde{\psi}_m^m(k; \mu_0)}{g_m^m(z)} + T(z, \mu; \mu_0) \end{aligned} \quad (104)$$

where

$$T(z, \mu; \mu_0) \equiv \sum_{l=m}^{\infty} \frac{(l-m)!}{(l+m)!} \frac{2l+1}{2} \chi_l^m(z, \mu_0) P_l^m(\mu) \quad (105)$$

and define the truncated version

$$\tilde{\psi}_L^m(k, \mu; \mu_0; N) = \phi_L^m(z, \mu; N) \frac{\tilde{\psi}_m^m(k; \mu_0)}{g_m^m(z)} + T(z, \mu; \mu_0; N) \quad (106)$$

where

$$T(z, \mu; \mu_0; N) \equiv \sum_{l=m}^N \frac{(l-m)!}{(l+m)!} \frac{2l+1}{2} \chi_l^m(z, \mu_0) P_l^m(\mu) \quad (107)$$

Ganapol [15, 16] proved that $T(z, \mu; \mu_0; N)$ is an analytic function in the case $m = 0$ and it would greatly simplify the Fourier transform inversion. For $m > 0$, we find that $g_l^m(z)$ and $\rho_l^m(z)$ are analytic functions, so $\chi_l^m(z, \mu_0)$ must be analytic too, and then $T(z, \mu; \mu_0; N)$. That is to say $T(z, \mu; \mu_0; N)$ will make no contributions to the Fourier transform inversion. Then we will focus on the first Legendre moment in Eq. (106).

From Eq. (51), we define

$$S(z, \mu_0) \equiv \varpi z P_m^m(z) \sum_{l=m}^L \beta_l^m Q_l^m(z) \chi_l^m(z, \mu_0) \quad (108)$$

Substituting Eq. (49) into Eq. (108), we get

$$\begin{aligned} S(z, \mu_0) &= (2m)! \varpi z P_m^m(z) \\ &\times \sum_{l=m}^L \beta_l^m Q_l^m(z) \sum_{j=m}^L \frac{(j-m)!}{(j+m)!} [\rho_j^m(z) g_l^m(z) - g_j^m(z) \rho_l^m(z)] S_j(\mu_0) \end{aligned} \quad (109)$$

Interchanging the sum sequence in above equation, we get

$$\begin{aligned} S(z, \mu_0) &= (2m)! P_m^m(z) \sum_{j=m}^L \frac{(j-m)!}{(j+m)!} S_j(\mu_0) \\ &\times \left\{ \begin{aligned} &\rho_j^m(z) \left[\varpi z \sum_{l=m}^L \beta_l^m g_l^m(z) Q_l^m(z) - \varpi z \sum_{l=m}^{j-1} \beta_l^m g_l^m(z) Q_l^m(z) \right] \\ &- g_j^m(z) \left[\varpi z \sum_{l=m}^L \beta_l^m \rho_l^m(z) Q_l^m(z) - \varpi z \sum_{l=m}^{j-1} \beta_l^m \rho_l^m(z) Q_l^m(z) \right] \end{aligned} \right\} \quad (110) \end{aligned}$$

From Eq. (16) and Eq. (18), we get

$$\begin{aligned} S(z, \mu_0) &= (2m)! P_m^m(z) \\ &\times \sum_{j=m}^L \frac{(j-m)!}{(j+m)!} \left\{ \begin{aligned} &\rho_j^m(z) [\Lambda_{j-1}^m(z) - \Lambda_L^m(z)] \\ &- g_j^m(z) [\gamma_{j-1}^m(z) - \gamma_L^m(z)] \end{aligned} \right\} S_j(\mu_0) \end{aligned} \quad (111)$$

Substituting $S_l^m(\mu_0) = (2l+1) P_l^m(\mu_0) e^{-im\varphi_0}$ into Eq. (111), we get

$$\begin{aligned}
S(z, \mu_0) &= (2m)! P_m^m(z) e^{-im\phi_0} \\
&\times \left\{ \sum_{j=m}^L \frac{(j-m)!}{(j+m)!} (2j+1) P_j^m(\mu_0) [\rho_j^m(z) \Lambda_{j-1}^m(z) - g_j^m(z) \gamma_{j-1}^m(z)] \right. \\
&\quad - \Lambda_L^m(z) \sum_{j=m}^L \frac{(j-m)!}{(j+m)!} (2j+1) \rho_j^m(z) P_j^m(\mu_0) \\
&\quad \left. + \gamma_L^m(z) \sum_{j=m}^L \frac{(j-m)!}{(j+m)!} (2j+1) g_j^m(z) P_j^m(\mu_0) \right\}
\end{aligned} \tag{112}$$

From Eq. (63) and Eq. (95), we get

$$\rho_j^m(z) \Lambda_{j-1}^m(z) - g_j^m(z) \gamma_{j-1}^m(z) = -\frac{z}{(2m)!} Q_j^m(z) \tag{113}$$

So

$$\begin{aligned}
&\sum_{j=m}^L \frac{(j-m)!}{(j+m)!} (2j+1) P_j^m(\mu_0) [\rho_j^m(z) \Lambda_{j-1}^m(z) - g_j^m(z) \gamma_{j-1}^m(z)] \\
&= -\frac{2z}{(2m)!} \sum_{j=m}^L \frac{(j-m)!}{(j+m)!} \frac{2j+1}{2} P_j^m(\mu_0) Q_j^m(z) \\
&= -\frac{2z}{(2m)!} \left[q_L^m(z, \mu_0) + \frac{1}{2} \frac{(1-\mu_0^2)^{m/2} (z^2-1)^{-m/2}}{z-\mu_0} \right]
\end{aligned} \tag{114}$$

From the definitions of the truncated version of GSEs Eq. (56) and Eq. (91), we get the relations

$$\begin{aligned}
\phi_L(z, \mu; L) &= \phi_L(z, \mu; N) \\
&\quad + \Lambda_L^m(z) [\delta_L^m(z, \mu) - \delta_N^m(z, \mu)] + \psi_L^m(z) [q_L^m(z, \mu) - q_N^m(z, \mu)]
\end{aligned} \tag{115}$$

$$\begin{aligned}
\Theta_L(z, \mu; L) &= \Theta_L(z, \mu; N) \\
&\quad + \gamma_L^m(z) [\delta_L^m(z, \mu) - \delta_N^m(z, \mu)] + \theta_L^m(z) [q_L^m(z, \mu) - q_N^m(z, \mu)]
\end{aligned} \tag{116}$$

So applying Eq. (101), we get

$$\begin{aligned}
S(z, \mu_0) &= (2m)! P_m^m(z) e^{-im\phi_0} \\
&\times \left\{ 2 \left[\gamma_L^m(z) \phi_L^m(z, \mu_0; N) - \Lambda_L^m(z) \Theta_L^m(z, \mu_0; N) - \frac{z}{(2m)!} q_N^m(z, \mu_0) \right] \right. \\
&\quad \left. - \frac{z}{(2m)!} \frac{(1-\mu_0^2)^{m/2} (z^2-1)^{-m/2}}{z-\mu_0} \right\}
\end{aligned} \tag{117}$$

Substituting above equation into Eq. (51), we get

$$\begin{aligned} \tilde{\psi}_m^m(k; \mu_0) &= \frac{2(2m)! P_m^m(z) e^{-im\phi_0}}{\Lambda_*^m(z)} \\ &\times \left[\gamma_L^m(z) \phi_L^m(z, \mu_0; N) - \Lambda_L^m(z) \Theta_L^m(z, \mu_0; N) - \frac{z}{(2m)!} q_N^m(z, \mu_0) \right] \end{aligned} \quad (118)$$

Different from the case $m = 0$, from Eq. (52) and Eq. (88) we get

$$\Lambda_*^m(z) = \Lambda_L^m(z) (z^2 - 1)^{m/2} \quad (119)$$

So

$$\begin{aligned} \tilde{\psi}_m^m(k; \mu_0) &= 2(2m)!(2m-1)!! e^{-im\phi_0} \\ &\times \left[\frac{\gamma_L^m(z)}{\Lambda_L^m(z)} \phi_L^m(z, \mu_0; N) - \Theta_L^m(z, \mu_0; N) - \frac{1}{\Lambda_L^m(z)} \frac{z}{(2m)!} q_N^m(z, \mu_0) \right] \end{aligned} \quad (120)$$

Thus, from Eq. (103) and Eq. (106), we get

$$\begin{aligned} \tilde{\psi}_L^m(k, \mu; \mu_0; N) &= 2(2m)! e^{-im\phi_0} \\ &\times \left\{ \frac{\phi_L^m(z, \mu_0; N) \phi_L^m(z, \mu; N)}{\varpi \phi_L^m(z, z)} \left[\frac{1}{(2m)!} \frac{1}{\Lambda_L^m(z)} + \varpi h_L^m(z, z) - \frac{(1-z^2)^{m/2}}{(2m)!} \right] \right. \\ &\quad \left. - \frac{1}{\Lambda_L^m(z)} \frac{z}{(2m)!} q_N^m(z, \mu_0) \phi_L^m(z, \mu; N) \right\} \\ &- H(z, \mu; \mu_0; N) \end{aligned} \quad (121)$$

where

$$H(z, \mu; \mu_0; N) \equiv 2(2m)! e^{-im\phi_0} \phi_L^m(z, \mu; N) \Theta_L^m(z, \mu_0; N) - T(z, \mu; \mu_0; N) \quad (122)$$

It is obvious that $H(z, \mu; \mu_0; N)$ is an analytic function and make no contributions to Fourier transform inversion. So the remaining task is to apply Fourier transform inversion to the expression in the brace of Eq. (121).

5. Fourier transform inversion

Applying Cauchy's integral theorem [24] to Fourier transform inversion is a popular and useful technique. To use the Cauchy's integral formula, we must construct a contour which includes the real integral on the real axis specific to Fourier transform inversion in the complex z -plane or k -plane. However, because $\Lambda_L^m(z)$ and $q_N^m(z, \mu_0)$ are holomorphic functions in the complex z -plane excluding a cut line -1 to 1 on the real axis, the cut line must be mapped to imaginary axis in the complex k -plane both in the upper half space and in the lower half space. Considering the symmetric radiance distribution about τ , from Eq. (25) we build the closed contour in the upper half space of the complex k -plane for $\tau \geq 0$ (shown in Fig. 1) and the contour integral:

$$I_L^m(\tau, \mu; \mu_0; N) = \lim_{\substack{R \rightarrow \infty \\ \varepsilon \rightarrow 0}} \frac{1}{2\pi} \oint_C \tilde{\psi}_L^m(k, \mu; \mu_0; N) e^{ik\tau} dk \quad (123)$$

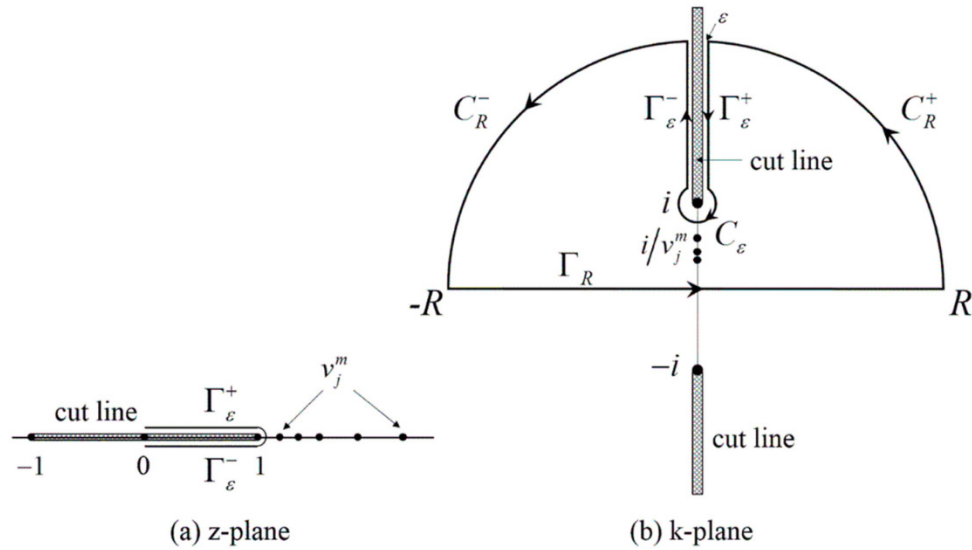


Fig. 1. Complex z-plane and k-plane

As shown in Fig. 1, the contour in k-plane includes six integral segments:

$$C \equiv \Gamma_R + C_R^+ + \Gamma_\varepsilon^+ + C_\varepsilon + \Gamma_\varepsilon^- + C_R^- \quad (124)$$

So the Fourier transform inversion integral is operating on the real axis Γ_R and we get the radiance component in position space

$$\psi^m(\tau, \mu; \mu_0; N) = I_L^m(\tau, \mu; \mu_0; N) - \lim_{\substack{R \rightarrow \infty \\ \varepsilon \rightarrow 0}} \frac{1}{2\pi} \int_{C-\Gamma_R} \tilde{\psi}_L^m(k, \mu; \mu_0; N) e^{ik\tau} dk \quad (125)$$

From Eq. (87) ~Eq. (89), as $R \rightarrow \infty$ or $|z| \rightarrow 0$, $\Lambda_L^m(z) \rightarrow 1$. $\phi_L^m(z, \mu_0; N)$, $\phi_L^m(z, \mu; N)$ and $\phi_L^m(z, z)$ are all finite polynomials and should be bounded. From the initial terms and three-term recurrence of $\rho_l^m(z)$, when $|z| \rightarrow 0$, $\rho_l^m(z) \rightarrow 0$, so from Eq. (93) $h_L^m(z, z) \rightarrow 0$. And, as $|z| \rightarrow 0$, from Eq. (74) and Eq. (79), $q_N^m(z, \mu_0)$ goes to a specific bounded value. So, when $|z| \rightarrow 0$, the first term in curly brackets contains in Eq. (121) should vanish, then by the Jordan's lemma [24], we will immediately conclude that as $R \rightarrow \infty$ the integrals over C_R^+ and C_R^- go to zero. As $\varepsilon \rightarrow 0$ the integral over C_ε goes to zero because $k = i$ or $z = 1$ is not the pole. So we only need to consider the contributions from the poles and the branch cut.

5.1 Application of the residue theorem

Before applying the residue theorem, we need to find out all the valid poles. From Eq. (121) the zeroes of $\Lambda_L^m(z)$ and $\phi_L^m(z, z)$ are candidates. From Eq. (119) the zeroes of $\Lambda_L^m(z)$ are just the discrete eigenvalues in Case's method. Case [25] proved that they are real, simple, plus-minus paired, finite in number, and equal to or larger than 1 in magnitude. We assume that there are M_L^m the positive eigenvalues v_j^m ($j = 1, 2, \dots, M_L^m$) which satisfy

$$\Lambda_L^m(\pm v_j^m) = 0 \quad (126)$$

It is easy to verify that the zeroes of $\Lambda_L^m(z)$ are valid poles, but the zeroes of $\phi_L^m(z, z)$ are removable singularities. Ganapol [16] proved that is true in the case $m = 0$. In the case $m > 0$, the derivation is totally similar, so we will not reiterate here.

Since v_j^m are the poles in z -plane and $z = i/k$, corresponding poles in k -plane are $k_j^m = i/v_j^m$, ($j = 1, 2, \dots, M_L^m$), as shown in Fig. 1. The contour integral now can be evaluated by residue theorem

$$I_L^m(\tau, \mu; \mu_0; N) = 2\pi i \sum_{j=1}^{M_L^m} \text{Res} \left[\frac{1}{2\pi} \tilde{\psi}_L^m(k, \mu; \mu_0; N); k = k_j^m \right] e^{ik_j^m \tau} \quad (127)$$

For each k_j^m we get

$$\begin{aligned} I_{L,j}^m(\tau, \mu; \mu_0; N) &= \text{Res} \left[\tilde{\psi}_L^m(k, \mu; \mu_0; N); k = k_j^m \right] e^{ik_j^m \tau} \\ &= \lim_{k \rightarrow k_j^m} (k - k_j^m) \tilde{\psi}_L^m(k, \mu; \mu_0; N) e^{ik_j^m \tau} \end{aligned} \quad (128)$$

Substituting Eq. (121) into above equation and converting the complex variable k into corresponding z , we get

$$\begin{aligned} I_{L,j}^m(\tau, \mu; \mu_0; N) &= -i \frac{\phi_L^m(v_j^m, \mu_0; N) \phi_L^m(v_j^m, \mu; N)}{\tilde{N}_0^m(v_j^m)} e^{-im\phi_0} e^{-\tau/v_j^m} \\ &\quad + i \frac{q_N^m(v_j^m, \mu_0) \phi_L^m(v_j^m, \mu; N)}{E_0^m(v_j^m)} e^{-im\phi_0} e^{-\tau/v_j^m} \end{aligned} \quad (129)$$

where

$$\tilde{N}_0^m(v_j^m) = \frac{1}{2} \varpi(v_j^m)^2 \phi_L^m(v_j^m, v_j^m) \frac{d\Lambda_L^m(z)}{dz} \Big|_{z=v_j^m} \quad (130)$$

$$E_0^m(v_j^m) = \frac{1}{2} v_j^m \frac{d\Lambda_L^m(z)}{dz} \Big|_{z=v_j^m} \quad (131)$$

Substituting Eq. (129) into Eq. (127), we get the contributions from the poles

$$\begin{aligned} I_L^m(\tau, \mu; \mu_0; N) &= \sum_{j=1}^{M_L^m} \frac{\phi_L^m(v_j^m, \mu_0; N) \phi_L^m(v_j^m, \mu; N)}{\tilde{N}_0^m(v_j^m)} e^{-im\phi_0} e^{-\tau/v_j^m} \\ &\quad - \sum_{j=1}^{M_L^m} \frac{q_N^m(v_j^m, \mu_0) \phi_L^m(v_j^m, \mu; N)}{E_0^m(v_j^m)} e^{-im\phi_0} e^{-\tau/v_j^m} \end{aligned} \quad (132)$$

5.2 Integral over the branch cut

As shown in Fig. 1, the branch cut is $[i, i\infty)$ on the imaginary axis in the upper half plane. We define $k_{\pm} = \pm \varepsilon + iy$, $y \in [1, \infty)$ and k_+ represents the integral path right side of the branch cut and k_- the left side. So $z_{\pm} = i/k_{\pm}$ and the branch cut in k -plane changes to the branch cut $(0, 1]$ on the real axis in z -plane, as shown in Fig. 1 (a).

So we get

$$z_{\pm} = \frac{i}{k_{\pm}} = \frac{y \pm i\varepsilon}{\varepsilon^2 + y^2} = v \pm i\varepsilon + O(\varepsilon^2), v \in [0, 1] \quad (133)$$

The integral over double sides of the branch cut is

$$\begin{aligned} I_{L,branch}^m(\tau, \mu; \mu_0; N) &= -\frac{i}{2\pi} \lim_{\varepsilon \rightarrow 0} \left\{ \int_{i\varepsilon}^{1+i\varepsilon} \frac{1}{z_+^2} \tilde{\psi}_L^m(z_+, \mu; \mu_0; N) e^{-\tau/z_+} dz_+ \right. \\ &\quad \left. - \int_{-i\varepsilon}^{1-i\varepsilon} \frac{1}{z_-^2} \tilde{\psi}_L^m(z_-, \mu; \mu_0; N) e^{-\tau/z_-} dz_- \right\} \\ &= -\frac{i}{2\pi} \int_0^1 \frac{e^{-\tau/v}}{v^2} \left[\tilde{\psi}_L^m(z_+, \mu; \mu_0; N) - \tilde{\psi}_L^m(z_-, \mu; \mu_0; N) \right] dv \end{aligned} \quad (134)$$

From above derivation we only need to consider the terms in Eq. (121) which include $\Lambda_L^m(z)$ or $q_N^m(z, \mu_0)$, so we get

$$\begin{aligned} &\tilde{\psi}_L^m(z_+, \mu; \mu_0; N) - \tilde{\psi}_L^m(z_-, \mu; \mu_0; N) \\ &= 2e^{-im\varphi_0} \frac{\phi_L^m(v, \mu_0; N) \phi_L^m(v, \mu; N)}{\varpi \phi_L^m(v, v)} \left\{ \frac{1}{[\Lambda_L^m(v)]^+} - \frac{1}{[\Lambda_L^m(v)]^-} \right\} \\ &\quad - 2e^{-im\varphi_0} v \phi_L^m(v, \mu; N) \left\{ \frac{[q_N^m(v, \mu_0)]^+}{[\Lambda_L^m(v)]^+} - \frac{[q_N^m(v, \mu_0)]^-}{[\Lambda_L^m(v)]^-} \right\} \end{aligned} \quad (135)$$

From Eq. (80) and Eq. (84), we get

$$\frac{1}{[\Lambda_L^m(v)]^+} - \frac{1}{[\Lambda_L^m(v)]^-} = -\frac{i\pi \varpi v \phi_L^m(v, v)}{[\Lambda_L^m(v)]^+ [\Lambda_L^m(v)]^-} \quad (136)$$

From Eq. (80) and Eq. (82), we get

$$\frac{[q_N^m(v, \mu_0)]^+}{[\Lambda_L^m(v)]^+} - \frac{[q_N^m(v, \mu_0)]^-}{[\Lambda_L^m(v)]^-} = \frac{i\pi F_L^m(v, \mu)}{[\Lambda_L^m(v)]^+ [\Lambda_L^m(v)]^-} \quad (137)$$

where

$$F_L^m(v, \mu) = [\mathcal{P}\delta_N^m(v, \mu) - \delta(v - \mu)] \mathcal{P}\Lambda_L^m(v) - \psi_L^m(v) \mathcal{P}q_N^m(v, \mu) \quad (138)$$

So

$$\begin{aligned} &\tilde{\psi}_L^m(z_+, \mu; \mu_0; N) - \tilde{\psi}_L^m(z_-, \mu; \mu_0; N) \\ &= -\frac{2i\pi v \phi_L^m(v, \mu_0; N) \phi_L^m(v, \mu; N)}{[\Lambda_L^m(v)]^+ [\Lambda_L^m(v)]^-} e^{-im\varphi_0} \\ &\quad - \frac{2i\pi v \phi_L^m(v, \mu; N) F_L^m(v, \mu)}{[\Lambda_L^m(v)]^+ [\Lambda_L^m(v)]^-} e^{-im\varphi_0} \end{aligned} \quad (139)$$

So the contributions from branch cut is

$$\begin{aligned}
 I_{L,branch}^m(\tau, \mu; \mu_0; N) \\
 = - \int_0^1 \left[\frac{\phi_L^m(v, \mu_0; N) \phi_L^m(v, \mu; N)}{\tilde{N}^m(v)} + \frac{2i\pi v \phi_L^m(v, \mu; N) F_L^m(v, \mu)}{v \tilde{N}^m(v)} \right] e^{-\tau/v} e^{-im\phi_0} dv
 \end{aligned} \quad (140)$$

where

$$\tilde{N}^m(v) = v \left[\Lambda_L^m(v) \right]^+ \left[\Lambda_L^m(v) \right]^- \quad (141)$$

5.3 Fourier transform inversion solution

Thus, we can build the Fourier transform inversion solution in the position space from Eq. (125)

$$\begin{aligned}
 \psi^m(\tau, \mu; \mu_0; N) &= \sum_{j=1}^{M_L^m} \frac{\phi_L^m(v_j^m, \mu_0; N) \phi_L^m(v_j^m, \mu; N)}{\tilde{N}_0^m(v_j^m)} e^{-\tau/v_j^m} e^{-im\phi_0} \\
 &\quad - \sum_{j=1}^{M_L^m} \frac{q_N^m(v_j^m, \mu_0) \phi_L^m(v_j^m, \mu; N)}{E_0^m(v_j^m)} e^{-\tau/v_j^m} e^{-im\phi_0} \\
 &\quad + \int_0^1 \left[\frac{\phi_L^m(v, \mu_0; N) \phi_L^m(v, \mu; N)}{\tilde{N}^m(v)} + \frac{2i\pi v \phi_L^m(v, \mu; N) F_L^m(v, \mu)}{v \tilde{N}^m(v)} \right] e^{-\tau/v} e^{-im\phi_0} dv
 \end{aligned} \quad (142)$$

Since now z has become real, as $N \rightarrow \infty$, we get

$$\begin{aligned}
 \psi^m(\tau, \mu; \mu_0) &= \sum_{j=1}^{M_L^m} \frac{\phi_L^m(v_j^m, \mu_0) \phi_L^m(v_j^m, \mu)}{\tilde{N}_0^m(v_j^m)} e^{-\tau/v_j^m} e^{-im\phi_0} \\
 &\quad + \int_0^1 \frac{\phi_L^m(v, \mu_0) \phi_L^m(v, \mu)}{\tilde{N}^m(v)} e^{-\tau/v} e^{-im\phi_0} dv
 \end{aligned} \quad (143)$$

Now, the last task is to confirm the norms in Eq. (142) or Eq. (143) are equal to those norms in our first accompanying paper, though it is apparently in $m=0$ case. In fact, from Eq. (119), this is undoubted without much efforts due to the common zeroes.

So we can finally conclude that the Fourier transform solution is consistent with the Case's singular eigenfunctions solution. The contributions from the poles in Fourier transform inversion are equivalent to the contributions of discrete eigenvalues in Case's method, and correspondingly, the branch cut contribution is equivalent to that of the continuous spectrum.

6. Conclusion

Both the Case's singular eigenfunctions expansion method and Fourier transform method are applied to the light transport integro-differential equations in the infinite homogeneous tissue with m -dependent anisotropic scattering kernel, and the consistence is proven from the singular eigenfunctions and the final series solution. However, some remarks are worth to be mentioned.

The form of solutions for three-term recurrences may be the most valuable point and can be combined with other methods, such as P_N method [26, 27]. We can predicate that the general analytical solutions that consist of the general solution and particular solution may be

applied to more approaches to seek the solutions of light transport equations in one/two/three-dimension full space, half space, and slab space etc. with general scattering kernel, especially when Case's method is not suitable. These general solutions and particular solutions are usually all related to the two kinds of the associated Chandrasekhar polynomials, so the value of the polynomials of the second kind may be developed intensively.

During our derivation, the identities involving Chandrasekhar polynomials and Legendre polynomials are often confusing due to the disunity of the definitions of some polynomials, such as the associated Legendre polynomials of the second kind. It is noted that the three-term recurrences for the associated Legendre polynomials of the second kind may be different from those in other books [20–22] in the first term. As Ganapol [16] and other experts pointed out, some functions in the derivation, such as delta function $\delta(z-\mu)$ and

$\frac{1}{2} \frac{(1-\mu^2)^{m/2} (z^2-1)^{-m/2}}{z-\mu}$, are extended into the complex plane, but their behaviors seem

difficult to interpret in mathematical manner. So some rational mathematical tricks must be applied.

In our first accompanying paper, four evaluations methods of the two kinds of Chandrasekhar polynomials have already been benchmarked in a computational manner. In this paper, the analyticity of the two kinds of Chandrasekhar polynomials are discussed in a mathematical manner. Besides, the measurable standards or formulas for the stabilities in m-dependent case are needed to be established, the mathematical features are still required to be explored thoroughly.

Lastly, we are revisiting the theories of the light transport in tissue and making great efforts on seeking the analytical solutions and more effective methods to improve the biomedical applications.

Funding

National Major Special Program of Scientific Instrument & Equipment Development of China (No. 2012YQ160203); Talent Plan 3551 of Wuhan City [No. WuXinGuan-(2010) 210], the Technology Innovation Fund of the National Science and Technology Ministry of China (No. 08C26224211088).

Acknowledgments

The main results of our two papers were obtained by extending derivation and benchmarks for the azimuth-independent Chandrasekhar polynomials by B. D. Ganapol to the m-dependent case and adapting these theories to biological optics.

Disclosures

The authors declare that there are no conflicts of interest related to this article.